A Class of Optimum Digital Phase Locked Loops for the DSN Advanced Receiver

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This article presents a class of optimum digital filters for digital phase locked loops of the DSN Advanced Receiver. The filter minimizes a weighted combination of the variance of the random component of the phase error and the sum square of the deterministic dynamic component of phase error at the output of the numerically controlled oscillator (NCO). By varying the weighting coefficient over a suitable range of values, a wide set of filters are obtained such that, for any specified value of the equivalent loop-noise bandwidth, there corresponds a unique filter in this class. This filter thus has the property of having the best transient response over all possible filters of the same bandwidth and type. The optimum filters are also evaluated in terms of their gain margin for stability and their steady-state error performance.

I. Introduction

There has been an increasing interest (Refs. 1-4) in the study of digital phase locked loops. Such an interest emerges in part from the capability of digital technology which makes it possible to control the loop parameters accurately and even make these programmable and/or adaptive.

This article derives optimum filters for the digital phase locked loops for carrier, subcarrier and symbol timing recovery in the DSN Advanced Receiver. The loop is analyzed in the z-transform domain so as to arrive at a set of optimum digital filters for various input dynamics.

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The filter minimizes a weighted sum of the variance of the random component of the phase error and the sum square of the deterministic dynamic component of the phase error at the output of the numerically controlled oscillator (NCO). By varying the weighting coefficient over a suitable range of values, a wide set of filters is obtained such that, for any specified value of the equivalent loop noise bandwidth, there corresponds a unique filter in this class. This filter thus has the property of having the best transient response over all possible filters of the same bandwidth and type.

Three specific filter sets optimum for a phase step, phase ramp and the phase acceleration inputs are considered in some detail. The corresponding optimum filters are type I, II, and III, respectively. Due to the specific optimization index under consideration, the filter design also ensures that the deterministic dynamic component of the steady-state phase error is zero for all input phase dynamics with the order of the highest order nonzero derivative term smaller than the filter type.

The open loop transfer function (the product of the transfer functions of the filter and the NCO) possesses a multiple pole at $z = 1$, with its multiplicity equal to the order of the nonzero phase derivative plus one. The type of the filter is the multiplicity of the pole at $z = 1$.

II. Optimum Filter

Figure 1 depicts the block diagram of the basic digital phase locked loop (DPLL) and the z domain model of its linearized version. In the figure, $F(z)$ represents the transfer function of the digital filter and $N(z)$ is the transfer function of the NCO and the transport lag in the loop, due to the specific implementation. We use the $N(z)$ appropriate for the DSN Advanced Receiver (Ref. 2),

$$N(z) = \frac{KT(z+1)}{2z^2(z-1)} \tag{1}$$

where K is some constant and T is the sampling period. Such an NCO transfer function results because of the limited loop update rate $1/T$. Similar results are possible for other $N(z)$.

The input noise $\xi(t)$ is assumed to be a zero mean white Gaussian noise with two-sided spectral density $N_0/2$. The noise $\bar{n}_i(t)$ is the real part of the complex envelope of $\xi(t)$ and is also white Gaussian with spectral density N_0.

In Fig. 1(b) is given a linearized and discrete-time version of the DPLL wherein $\{n_i(k)\}$ represents a zero mean white Gaussian sequence with variance (N_0/A^2T). In this model $\hat{\Theta}$ may be decomposed into a sum of $\psi(k)$ and $n_0(k)$ where $\psi(k)$ represents the deterministic part of $\hat{\Theta}$ while $n_0(k)$ is the stochastic component. The optimum filter is derived by minimization of the following index (Refs. 4, 5, 6):

$$Q = E\,[n_0^2(k)] + \lambda \sum_k e^2(k) \tag{2}$$

with $e(k) = \Theta(k) - \psi(k)$ denoting the deterministic component of the phase error. The parameter λ is selected from the consideration of the loop noise bandwidth and the transient performance of the loop, which are functions of λ for the optimum solution. In the sequel, the parameter λ is expressed in terms of $\bar{N}_0$, T and an appropriate normalized parameter. Thus, for example, when the input phase is a step function of time, $\lambda = \bar{N}_0 Tq$ where q is the normalized parameter. The first term in the index Q can be expressed in terms of the loop filter $F(z)$ as

$$E\,[n_0^2(k)] = \frac{1}{2\pi j}\int_\Gamma \Phi_{n_0n_0}(z)\,\frac{dz}{z} \tag{3}$$

with $\Phi_{n_0n_0}(z)$ denoting the noise spectral density of $n_0(k)$. Now

$$\Phi_{n_0n_0}(z) = H(z)H(z^{-1})\,\Phi_{n_in_i}(z);$$

$$H(z) = \frac{F(z)N(z)}{1+F(z)N(z)}$$

$$\Phi_{n_in_i}(z) = \frac{N_0}{A^2T} = \frac{\bar{N}_0}{T}; \qquad \bar{N}_0 = N_0/A^2$$

Similarly, the second term in Eq. (2) can be evaluated in terms of the following contour integral (Γ denotes the unit circle),

$$\sum_k e^2(k) = \frac{1}{2\pi j}\int_\Gamma E(z)E(z^{-1})\,\frac{dz}{z}; \qquad E(z) = Z\,[e(k)]$$

where Z denotes z-transform. As $E(z) = (1-H(z))\,\Theta(z)$ (where $\Theta(z)$ denotes z-transform of $\theta(k)$), an equivalent expression for the sum of squared errors is

$$\sum_k e^2(k) = \frac{1}{2\pi j}\int_\Gamma (1-H(z))\,(1-H(z^{-1}))\,\Phi_{\theta\theta}(z)\,\frac{dz}{z};$$

$$\Phi_{\theta\theta}(z) = \Theta(z)\,\Theta(z^{-1}) \tag{4}$$

From Eqs. (3) and (4), the optimization index Q may be rewritten as,

$$Q = \frac{1}{2\pi j}\int_\Gamma [\lambda\Phi_{\theta\theta}(z) + P(z)\,W(z)\,W(z^{-1})$$

$$- \lambda W(z)\,N(z)\Phi_{\theta\theta}(z) - \lambda W(z^{-1})\,N(z^{-1})\,\Phi_{\theta\theta}(z)]\,\frac{dz}{z};$$

$$H(z) = W(z)\,N(z); \qquad P(z) = [\bar{N}_0T + \lambda\Phi_{\theta\theta}(z)]\,N(z)\,N(z^{-1}) \tag{5}$$

The optimum solution for $W(z)$ and thus $H(z)$ can then be obtained from the spectral factorization (Refs. 4, 5, 6) as

$$W_0(z) = \frac{z\left[\dfrac{\lambda N(z^{-1})\Phi_{\theta\theta}(z)}{zP^{-}(z)}\right]_{+}}{P^{+}(z)}; \qquad P(z) = P^{+}(z)P^{-}(z) \tag{6a}$$

In the above $P^{+}(z)$ denotes that factor of $P(z)$ which has all its poles and zeros inside the unit circle, and $[C(z)]_{+}$, for any rational function $C(z)$, represents that part of the partial fraction expansion of $C(z)$ whose poles are inside the unit circle.

The optimum loop filter $F(z)$ can then be obtained from $W_0(z)$ as

$$F(z) = \frac{W_0(z)}{1 - W_0(z)N(z)} \tag{6b}$$

III. Performance Considerations of Optimum Filters

The optimum filters of the previous section are evaluated in terms of various parameters of interest. The most significant parameters are the loop noise bandwidth, transient error performance, stability analysis and gain margin, and the steady state phase error for an input having a nonzero derivative of order equal to the filter type.

A. Loop Noise Bandwidth

In terms of the filter performance, it is of interest to evaluate the phase noise variance $E[n_0^2(k)]$ as the weighting parameter λ or its normalized version is varied. This variance is given by Eq. (3) and is usually expressed in terms of a normalized parameter B given by

$$B = \frac{1}{2\pi j}\int_{\Gamma} H(z)H(z^{-1})\frac{dz}{z} \tag{7}$$

The parameter B is termed normalized loop noise bandwidth and in terms of B, the oscillator phase noise variance is given by $(\bar{N}_0 B/T)$. In the PLL literature, a loop noise bandwidth B_L is also defined and equals (B/T) Hz.

B. Computation of B

From Ref. 7, the integral for B can be expressed in terms of the coefficients of $H(z)$. Let

$$H(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}; \qquad a_0 \neq 0$$

where the coefficients a_r, $0 < r \leqslant n$ and b_r, $0 \leqslant r \leqslant n$ are not necessarily nonzero. Then

$$\Omega v = u \tag{8}$$

where Ω is an $n \times n$ matrix and v and u are n-dimensional vectors with the elements of Ω and u being functions of the coefficients a's and b's. Moreover, the first element of v equals $a_0 B$; thus, B can be computed by solving the set of n linear equations (8). In this regards, it is of interest to note that the matrix Ω can be decomposed as,

$$\Omega = \Omega_{UT} + \Omega_{UH} - \bar{\Omega}$$

where Ω_{UT} and Ω_{UH} are upper triangular Toeplitz and "upper triangular" Hankel matrices respectively, i.e.,

$$\Omega_{UT} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}$$

$$\Omega_{UH} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3 & a_4 & \cdots & 0 \\ a_2 & a_3 & a_4 & a_5 & \cdots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_n & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{9}$$

and $\bar{\Omega}$ has its first row equal to $[a_0 \; a_1 \; a_2 \cdots a_n]$ with the remaining rows identically equal to zero. The elements of the vector u are expressed in terms of the autocorrelation function of the sequence b_i, i.e.,

$$u^T = [R_{bb}(0) \; 2R_{bb}(1) \cdots 2R_{bb}(n)]; \qquad R_{bb}(k) \triangleq \sum_{i=0}^{n} b_i b_{i+k}$$

With a slight modification, the solution for B can be expressed in the following form (possessing a rich structure for computational purposes),

$$B = \frac{2}{a_0} \times \text{first element of } \{(\Omega_{UT} + \Omega_{UH})^{-1} \bar{u}\}$$

$$\bar{u} \triangleq [R_{bb}(0) R_{bb}(1) \cdots R_{bb}(n)]^T \tag{10}$$

The specific structure of the matrix in Eq. (10) can be exploited in the fast computation of the bandwidth. Thus, when the dimension of the matrix $(\Omega_{UT} + \Omega_{UH})$ is high, sophisticated algorithms requiring order $n \log_2 n \log_2 n$ operations can be used for solving Eq. (10).

C. Transient Performance

The transient performance of the filter is evaluated in terms of the index

$$\sum_k e^2 (k)$$

where $e(k)$ is the deterministic component of the phase error. A smaller value of this index also implies a faster settling of the dynamic component of the phase error to zero. In principle, the value of this index can be reduced to an arbitrarily small value by choosing the value of the parameter λ sufficiently high. However, this would have the adverse effect of increasing the loop noise bandwidth and degrading the noise performance of the loop. The objective is thus to arrive at a compromise solution, as in the following section.

D. Stability Analysis and Gain Margin

In the preceding derivation of the optimum filter, it is assumed that the input signal amplitude A is constant (implicitly assumed to be 1). If the gain A is known then it can be taken into account by dividing the constant of $F(z)$ by A. However, in actual practice, there may be some uncertainty associated with A or A may be a slowly varying function of time. In such situations it is essential that the loop remain stable for a sufficiently large range of A.

In the subsequent section, using Jury's criterion, the upper and lower gain margins are evaluated for the closed loop stability of the loop. Such an evaluation is particularly important, since the filter derivation does not explicitly take into account such an index.

E. Steady State Error Due to Higher Order Dynamics

In practice, the signal may possess a component with the order of its highest nonzero derivative equal to or greater than the type of the filter. For example, the phase locked loop with a type III filter may have a nonzero phase jerk at its input. In this case, the deterministic component of the steady state phase error is nonzero and it may be of interest to evaluate the derived filter in terms of such steady state phase error ϕ_{ss}.

For analog phase locked loops (Refs. 5, 6) ϕ_{ss} is inversely proportional to B_L^i, where i denotes the filter type and B_L is the actual loop noise bandwidth in Hz. As $B_L = B/T$, a suitably normalized parameter $C_{\phi i}$ is given by $C_{\phi i} = (\phi_{ss})^{1/i} B/T$, where the order of the highest nonzero derivative of the input signal is assumed to be equal to i. The variation of $C_{\phi i}$ is studied in a subsequent section, as the normalized loop noise bandwidth B varies over the range of interest.

IV. Performance Evaluation of the Specific Filter Classes

In the following, we evaluate the parameters of the optimum filter $W_0(z)$ and the loop filter $F(z)$ as the parameter λ is varied for the NCO transfer function given by Eq. (1). The design of three specific classes of filters is considered to correspond to a phase step, frequency step and a frequency ramp input to the PLL. These filters are type I, II and III, respectively. For each class of filters, various design curves are obtained for the parameters and performance including the normalized design parameter, the optimum loop gain, the pole and zero locations of $F(z)$, the upper and lower gain margin for stability, and the steady state error constant $C_{\phi i}$, as functions of the normalized loop noise bandwidth, B.

A. Optimum Filter for Phase Step (Type I)

In this case, the input phase function $\theta(t) = u(t)$, the unit step function, and thus $\Theta(z) = (1 - z^{-1})^{-1}$. As shown in the appendix, the optimum filter $W_0(z)$ in this case is given by

$$W_0(z) = \frac{a+b+c}{KT} \left[\frac{(z-1)z}{(az^2 + bz + c)} \right]$$

where a, b and c must satisfy the set of following equations

$$\left.\begin{aligned} ac &= -1 \\ ab + bc &= q \,; \qquad q = \lambda/\bar{N}_0 T \\ a^2 + b^2 + c^2 &= 2(1+q) \end{aligned}\right\} \tag{11}$$

The optimum loop filter $F(z)$ can then be obtained from $W_0(z)$ in a straightforward manner and after a few manipulations can be written as

$$F(z) = \frac{2(a+b+c)}{KT}$$

$$\times \left\{ \frac{z^2}{2az^2 + 2(b+a)z + (a+b+c)} \right\} \qquad (12)$$

From the computations of the pole-zero locations of $F(z)$, it turns out that the filter $F(z)$ possesses a pole at $z = -1$, i.e.,

$$az^2 + bz + c = (\bar{a}z + \bar{b})(z+1) \qquad (13a)$$

where $\bar{a}, \bar{b}$ satisfy the following

$$\begin{aligned} \bar{a}\bar{b} &= -1 \\ \bar{a}^2 + \bar{b}^2 &= 2 + q \end{aligned} \qquad (13b)$$

Even though the pole at -1 is optimal, it is desirable to avoid a pole-zero cancellation at $z = -1$ so as to avoid the practical problem of imperfect pole cancellation and hence instability. Therefore, the filter coefficients are modified so as to shift the pole at the unit circle to $z = -0.9$ (say). Thus the factor $(z+1)$ in Eq. (13a) is replaced by $(z + 0.9)$ with corresponding changes in Eq. (13b).

From Eqs. (1) and (12), the open loop transfer function for the type I PLL is given by,

$$F(z)N(z) = \bar{K}_{opt} \frac{(z+1)}{(z-1)\left\{z^2 + \left(\frac{b+a}{a}\right)z + \left(\frac{a+b+c}{2a}\right)\right\}};$$

$$\bar{K}_{opt} = 0.95 \left(\frac{\bar{a}+\bar{b}}{\bar{a}}\right) \qquad (14)$$

Moreover, the corresponding closed loop transfer function of the PLL is given by

$$H(z) = \frac{a+b+c}{2} \quad \frac{z+1}{z(az^2+bz+c)}$$

In this case, the normalized loop noise bandwidth B as evaluated from Eq. (7) has the following closed-form expression.

$$B = \frac{a+b+c}{2(a-c)} \qquad (15)$$

1. **Filter parameters.** In the practice of phase locked receiver design, it is customary to treat the normalized loop noise bandwidth B as an independent parameter and then express various other loop parameters and the performance indices as functions of B. In the case of digital PLL, closed form expressions of this form cannot possibly be derived. Thus to obtain such relationships numerically, we evaluate $\bar{a}, \bar{b}$ first as a function of q from the modified version of Eq. (13b). (Actually for the case of computations, b and q are calculated as a function of $\bar{a}$ which can have values strictly greater than 1.) In this case, from the modified version of Eq. (13a), we also note that $a = \bar{a}$, $b = \bar{b} + 0.9\bar{a}$, $c = 0.9\bar{b}$.

In Fig. 2 is plotted the value of the parameter q which is required to obtain the normalized bandwidth B for the type I PLL. As is apparent from the figure, corresponding to the range of B between 0.001 and 0.95, the parameter q varies over approximately 7 orders of magnitudes (10^{-5} to 10^2). To design an optimum type I filter of bandwidth B, one can determine the loop gain from Fig. 3 and the pole locations from Fig. 4.

2. **Stability**. From stability theory, the closed loop system corresponding to the open loop transfer function of Eq. (14) remains stable if $\bar{K}_{opt}$ is replaced by any positive $\bar{K} \leqslant \bar{K}_{opt}$. However, if $\bar{K}$ is greater than $\bar{K}_{opt}$, then the closed loop system may become unstable. The maximum value of $\bar{K}$ for closed loop stability, denoted $\bar{K}_{max}$ is expressed in terms of the upper gain margin $G_u \triangleq 20 \log (\bar{K}_{max} | \bar{K}_{opt})$ (dB) and is plotted versus B in Fig. 5. As may be inferred from the figure, for a normalized loop noise bandwidth B less than or equal to 0.1, the optimum filter yields an upper gain margin of 16 dB or higher.

In Fig. 6(a) is plotted the root locus diagram for a typical filter of type 1, corresponding to a nominal bandwidth of 0.0965. As the loop gain is varied around the optimum gain $\bar{K}_{opt}$, the normalized bandwidth B also varies around its nominal value as depicted in Fig. 6(b).

3. **Transient error performance**. Figure 7 plots the integral square error, given by Eq. (4) as a function of normalized bandwidth. Over the bandwidth range of interest, the error varies approximately by two orders of magnitude. Note that some suboptimality is introduced due to modification of the pole at $z = -1$.

4. **Steady state error due to phase ramp.** The steady state phase error is given by

$$\phi_{ss} = \lim_{z\to 1} \left(\frac{z-1}{z}\right) \frac{1}{1+F(z)N(z)} \Theta(z)$$

$$\Theta(z) = T \frac{z}{(z-1)^2}$$

or,

$$\phi_{ss} = \lim_{z\to 1} \frac{T}{(z-1)F(z)N(z)}$$

For the optimum filter (Eq. [14]) ϕ_{ss} is given by

$$\phi_{ss} = T \frac{5a+3b+c}{2(a+b+c)} = C_{\phi 1}/B_L$$

Figure 8 plots the normalized value of ϕ_{ss} i.e., $C_{\phi 1}$ as a function of B. The radian phase error for radian center frequency ω_0 and the spacecraft velocity v is $(\omega_0 v/c)\,(C_{\phi 1}/B_L)$ where c is the speed of light.

In Fig. 9 is plotted the location of the closed loop pole as a function of B. As can be inferred from the figure, increasing the value of B results in the movement of the pole towards the origin, thus resulting in faster transient response.

5. **Type I summary.** Table 1 lists some of the parameters of six typical filter corresponding to different values of q for quick reference.

B. Optimum Filter for Frequency Step (Type II)

In this case, the input phase function $\theta(t) = t\,u(t)$ and thus

$$\Theta(z) = \frac{Tz}{(z-1)^2}.$$

The optimum filter $W_0(z)$ is derived in the appendix and is given by

$$W_0(z) = \frac{2}{KT} \frac{(h_0 z - h_1)(z-1)z}{4(az^3+bz^2+cz+d)}$$

where

$$h_0 = (7a+5b+3c+d)$$

$$h_1 = (5a+3b+c-d)$$

and the parameters a, b, c, d are the solutions of the following set of nonlinear equations.

$$\left.\begin{aligned} ad &= 1 \\ ac+bd &= -2 \\ ab+bc+cd &= r-1 \\ a^2+b^2+c^2+d^2 &= 2r+4 \\ r &= \frac{\lambda T}{N_0} \end{aligned}\right\} \qquad (16)$$

Using the identity that $(a+b+c+d)^2 = 4r$, and after a few algebraic manipulations, the corresponding loop filter is given by

$$F(z) = \frac{2}{KT} \times \frac{(h_0 z - h_1)z^2}{\{4az^2 + (8a+4b)z + (5a+3b+c-d)\}(z-1)} \qquad (17)$$

As in the case of phase-step, there is a pole-zero cancellation at $z = -1$ in the filter $F(z)$, i.e., the coefficients a, b, c, d satisfy the following identity

$$az^3+bz^2+cz+d = (z+1)(\bar{a}z^2+\bar{b}z+\bar{c})$$

where $\bar{a}$, $\bar{b}$, and $\bar{c}$ satisfy the following set of equations

$$\left.\begin{aligned} \overline{ac} &= 1 \\ \overline{ab}+\overline{bc} &= -4 \\ \bar{a}^2+\bar{b}^2+\bar{c}^2 &= 6+r \end{aligned}\right\} \qquad (18)$$

The modified coefficients a, b, c, d are then obtained according to the following relation

$$(az^3+bz^2+cz+d) = (z+0.9)(\bar{a}z^2+\bar{b}z+\bar{c})$$

1. **Filter parameters.** Figure 10 plots the normalized bandwidth B versus the optimizations parameter r. Figures 11, 12, and 13 plot the loop gain and those zeros and poles of the optimum filter whose locations depend upon B. A comparison of these plots with the corresponding plots of type I filter indicates that for low values of B, the pole locations are nearly the same for both types. Similar remarks apply with respect to loop gain.

2. **Stability**. The upper and lower gain margins are plotted versus B in Fig. 14. An upper gain margin of 10 dB or higher is obtained for all B less than 0.5, and the lower gain margin is very large for all B less than 1.

3. **Transient response.** The transient response is given in Fig. 15. Compared to the type I filter, the transient error varies over a much higher range – approximately six orders of magnitude.

4. **Steady state error to frequency ramp.** In Fig. 16 is plotted the normalized steady state phase error constant $C_{\phi 2} = (\phi_{ss}^{1/2}\ B/T)$ for a unit frequency ramp input to the PLL. A limiting value of just over 1 is approached for B less than 0.1. The radian phase error for radian center frequency ω_0 is $(\omega_0\ a/c)\ (C_{\phi 2}/B_L)^2$ where a is acceleration (in m/s^2 for c in m/s).

C. Optimum Filter for Frequency Ramp (Type III)

In this case

$$\Theta(z) = \frac{T^2 z(z+1)}{(z-1)^3}$$

and the optimum filter $W_0(z)$ is given by (the details similar to the first two cases and omitted):

$$W_0(z) = \frac{2}{KT}\ \frac{\{\widetilde{C}z^2 + (\widetilde{B} - 2\widetilde{C})z + (\widetilde{A} + \widetilde{C} - \widetilde{B})\}z(z-1)}{16(az^4 + bz^3 + cz^2 + dz + e)} \tag{19}$$

and the loop filter transfer function $F(z)$ is given by

$$F(z) = \frac{2}{KT}$$

$$\times \frac{\{\widetilde{C}z^2 + (\widetilde{B} - 2\widetilde{C})z + (\widetilde{A} + \widetilde{C} - \widetilde{B})\}z^2}{(z-1)^2\ \{16az^2 + 16(3a+b)z + [16(6a+3b+c) - \widetilde{C}]\}}$$

where

$$\left.\begin{aligned}
\widetilde{A} &= 8(a+b+c+d+e) \\
\widetilde{B} &= 4(9a+7b+5c+3d+e) \\
\widetilde{C} &= 2\left\{\bar{Q}_2 - \frac{2\bar{Q}_1\ (b+2c+3d+4e)}{(a+b+c+d+e)}\right\} \\
\bar{Q}_1 &= (9a+7b+5c+3d+e) \\
\bar{Q}_2 &= (33a+33b+29c+21d+9e)
\end{aligned}\right\} \tag{20}$$

The coefficients a, b, c, d and e are obtained by equating the coefficients of various powers of z on both sides of

$$az^4 + bz^3 + cz^2 + dz + e = (z+0.9)(\bar{a}z^3 + \bar{b}z^2 + \bar{c}z + \bar{d}) \tag{21}$$

and $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are obtained as a solution of the

$$\left.\begin{aligned}
\bar{a}\bar{d} &= -1 \\
\bar{a}\bar{c} + \bar{b}\bar{d} &= 6 \\
\bar{a}\bar{b} + \bar{b}\bar{c} + \bar{c}\bar{d} &= -(15-s) \\
\bar{a}^2 + \bar{b}^2 + \bar{c}^2 + \bar{d}^2 &= 20 + 2s;\quad s = \frac{\lambda T^3}{N_0}
\end{aligned}\right\} \tag{22}$$

1. **Parameters of type III filter.** The parameters of the optimum filter, i.e., the loop gain, location of zeros and poles (those depending upon B) are quite close to those for type I and II filters for $B \leqslant 0.5$. For $B > 0.5$ however, there is significant difference in the location of poles. For space limitations the plots of these parameters versus B are omitted.

2. **Performance.** The performance indices of the filters are plotted in Figs. 17 through 19. As may be inferred from Fig. 17, both upper and lower gain margins of 10 dB or higher are achieved for normal regions of operation. The dynamic error (Fig. 18) has an extremely large range (about 12 orders of magnitude) for the range of B of interest. This suggests a programmable implementation of the PLL for fast acquisition as is discussed below. In Fig. 19 is plotted the normalized steady state phase error constant $C_{\phi 3} = \phi_{ss}^{1/3}\ B/T$. For low values of B, this has a value of approximately 1.5. Radian phase error is $(\omega_0 j/c)\,(C_{\phi 3}/B_L)^3$ where j is jerk (m/s^3).

V. Programmable and Adaptive Implementations

From the performance analysis of the optimum filters the following adaptive implementations are suggested.

The phase noise variance at the output of the NCO is given by $(N_0 B/A^2 T)$. Thus in those situations where (A^2/N_0) (the input signal power to noise spectral density) is slowly varying with time, in order to maintain the phase noise variance within specified limit, say V_{max}, a value is obtained from $B = V_{max}\ T(A^2/N_0)$. Then from the plots of the filter parameters (stored in the memory in the form of tables versus B) an optimum filter corresponding to B is obtained. This filter then also has the best transient response consistent with the desired phase variance.

Because the transient error varies over an extremely large range (about 12 orders of magnitude for type III filter), a rapid acquisition can be attained in the following manner. Starting with a sufficiently high value of B (thus very rapid settling of the loop), the loop parameters are adjusted, at an interval of several times the dominant time constant of the filter, to the values corresponding to successive lower values of B until the desired value of B is achieved.

As the filter poles location are not sensitive to the input dynamics, a serial implementation of the filter is suggested. Acquisition can be accomplished with a type II loop, which has a good transient response, and then the type can be increased by type III by adding a pole at $z = 1$. The feasibility of acquisition with the type II loop depends upon an estimate (upper bound) of the magnitude of the acceleration.

VI. Conclusions

The performance of a class of optimum filters for three different input phase dynamics has been evaluated. The filters achieve minimum transient error for any loop bandwidth. The optimum filter is such that the open loop transfer function has multiple poles at $z = 1$ with the multiplicity m equal to one plus the number of nonzero derivatives ($t > 0$) terms of the input. In addition the filter has an optimum compensator with its denominator polynomial of degree 2 and the degree of numerator polynomial equal to m.

The parameters of the optimum compensator have been obtained as a function of an optimizing parameter q. Increasing the value of q has the effect of weighing more heavily the deterministic component of the phase error. This has the effect of achieving smaller transient error by increasing the loop gain, placing the open-loop poles closer to the unit circle and the filter zeros close to the origin. Alternatively, higher values of q bring the poles of the closed-loop system closer to the origin. Such a behavior of the system may be very desirable during the acquisition phase of the loop. Thus, in an adaptive implementation, a loop filter corresponding to high values of q may be used during the acquisition phase. During the tracking phase then a filter corresponding to a lower value of q may be switched in.

The filters designed on this basis also have good margin against possible variations in the received signal power level.

References

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Table 1. Performance of optimum filters (type I) corresponding to different values of the parameter q

q	Optimum Loop Gain $\bar{K}$ Optimum	Optimum Loop Noise BW	Loop Transfer Function	Range of $\bar{K}$ for Stability	Gain Margin in dB	Bandwidth Over ±12 dB Gain Variation
0.0364	0.1649	0.0965	$\dfrac{z+1}{(z-1)(z^2+1.0736z+0.1649)}$	0.002 – 1.091	–38, 16.4	0.018 – 0.945
0.1344	0.2903	0.1786	$\dfrac{z+1}{(z-1)(z^2+1.2056z+0.2903)}$	0.002 – 1.204	–43, 12.35	0.034 – 1.2
0.4702	0.4653	0.3189	$\dfrac{z+1}{(z-1)(z^2+1.390z+0.4653)}$	0.002 – 1.380	–47, 9.66	0.11 – 1.35*
1.548	0.6568	0.5140	$\dfrac{z+1}{(z-1)(z^2+1.5914z+0.6568)}$	0.002 – 1.60	–50, 7.73	0.17 – 3.8*
10.33	0.8724	0.8127	$\dfrac{z+1}{(z-1)(z^2+1.8184z+0.8724)}$	0.002 – 1.805	–52.8, 6.31	0.25 – 18.*
180.2	0.9448	0.9401	$\dfrac{z+1}{(z-1)(z^2+1.8945z+0.9448)}$	0.002 – 1.88	–53.5, 6.0	0.30 – 38.*

*Entries correspond to ±6 dB gain variation.

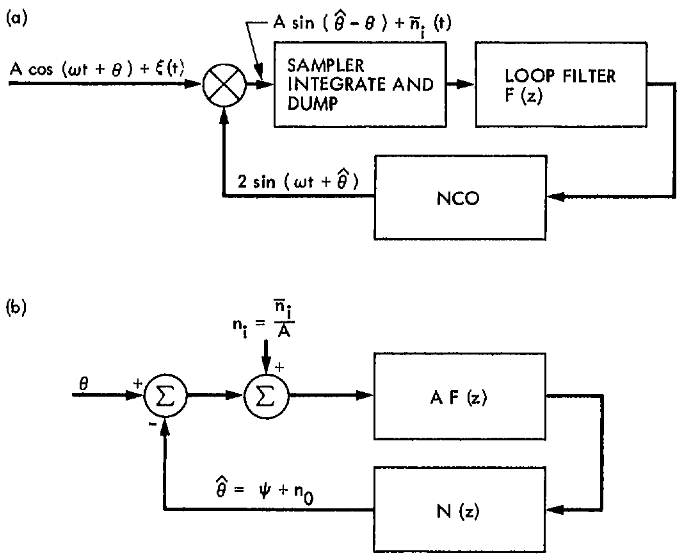

Fig. 1. Digital phase locked loop: (a) basic phase locked loop, (b) linearized model

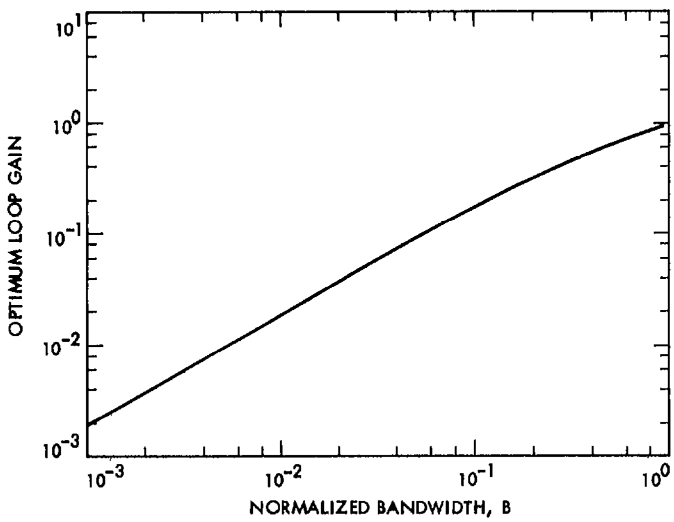

Fig. 3. Optimum loop gain vs normalized bandwidth for type I filter

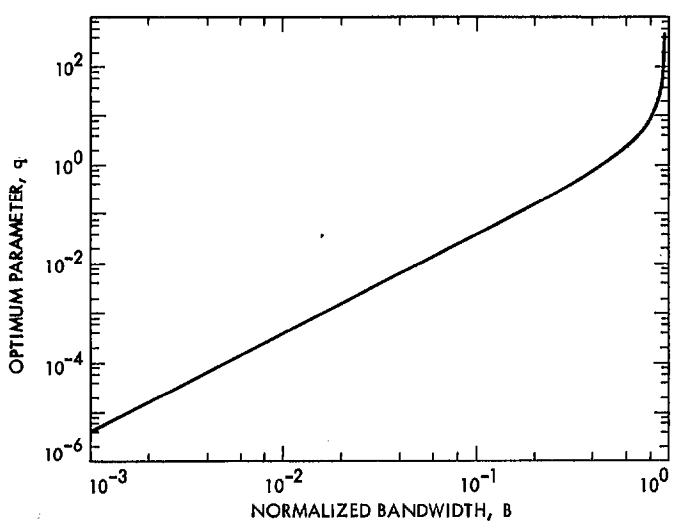

Fig. 2. Optimum parameter *q* for loop filter of type I

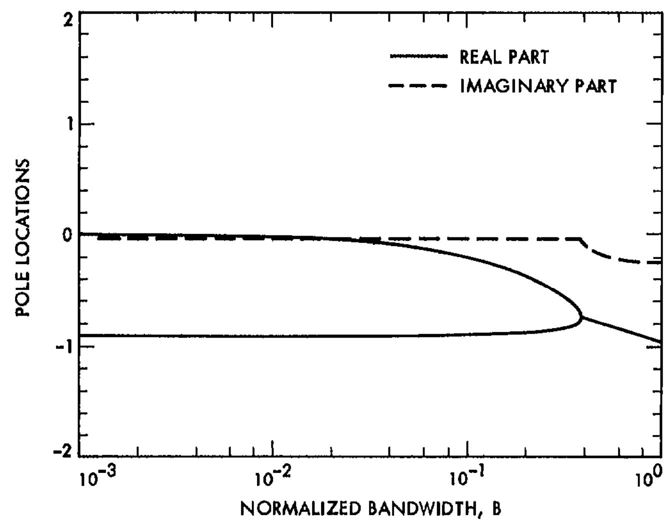

Fig. 4. Pole locations of the optimum loop filter of type I

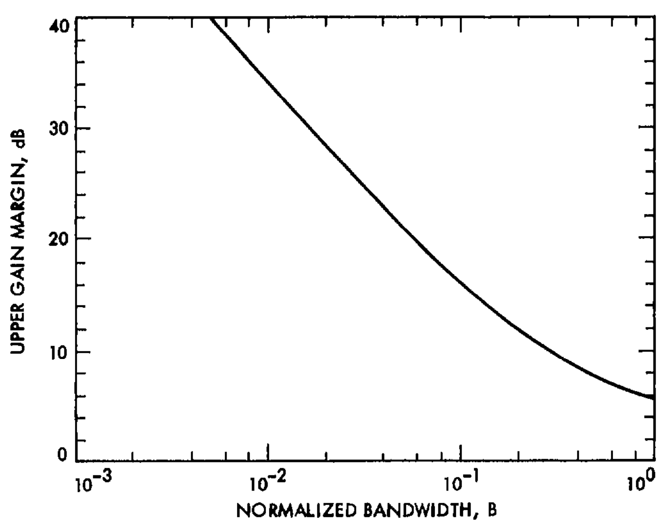

Fig. 5. Upper gain margin vs normalized bandwidth

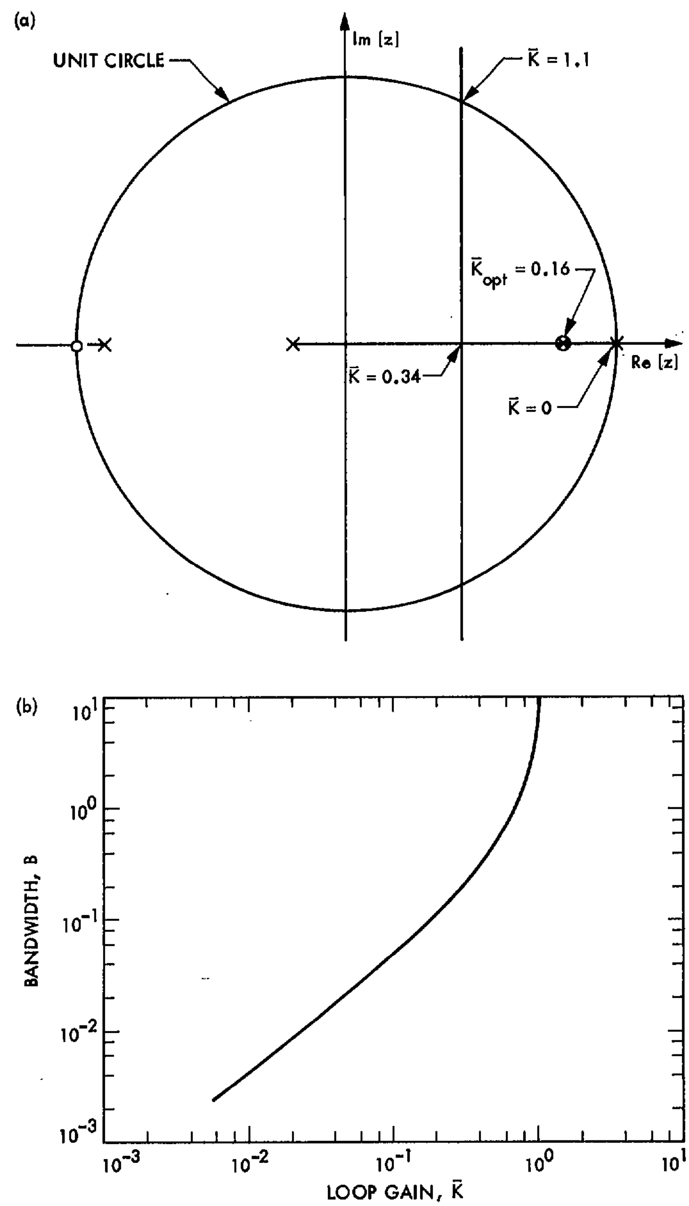

Fig. 6. Performance of type I filter vs loop gain: (a) root locus diagram for type I PLL, (b) loop noise bandwidth vs loop gain $\bar{K}$ for type I filter (nominal BW = 0.0965)

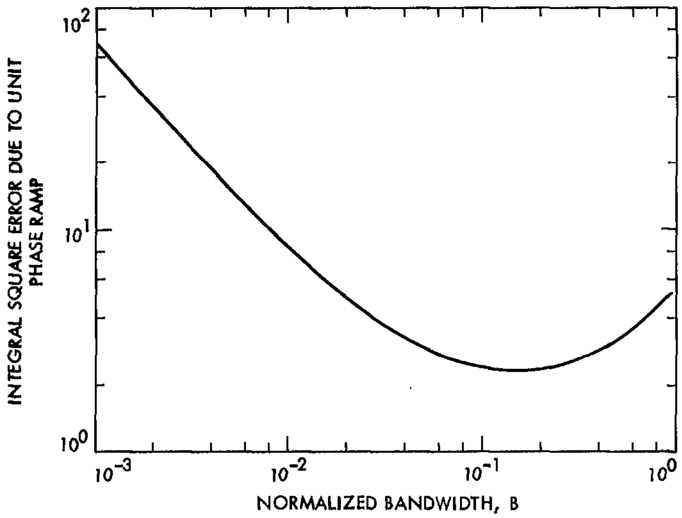

Fig. 7. Transient error performance of type I filter

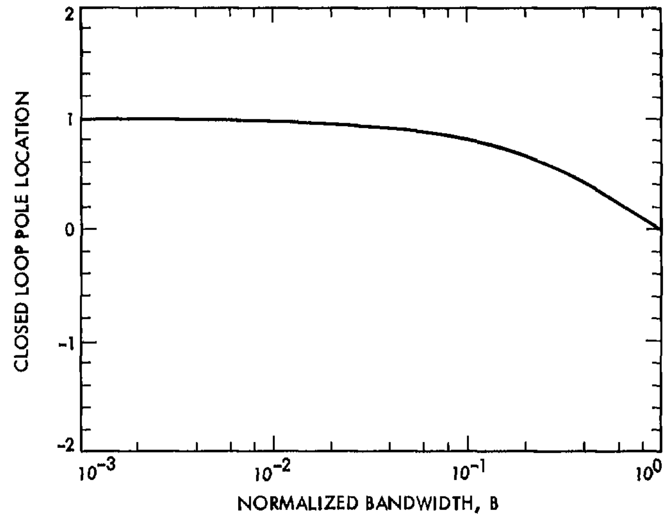

Fig. 9. Closed loop pole location of type I PLL

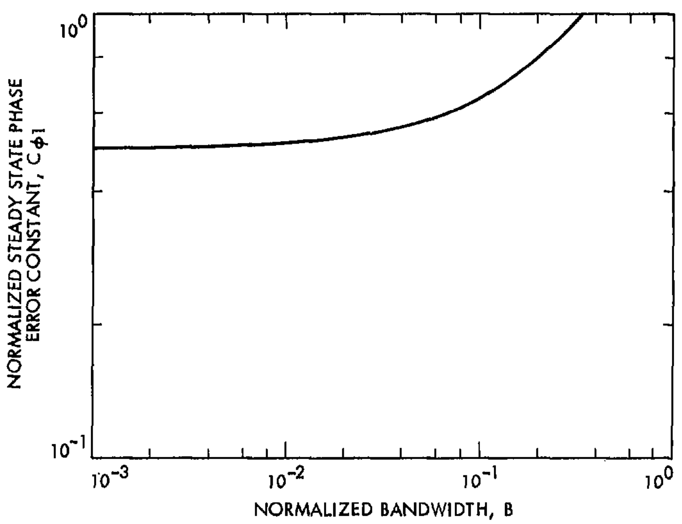

Fig. 8. Steady state error performance of type I filter

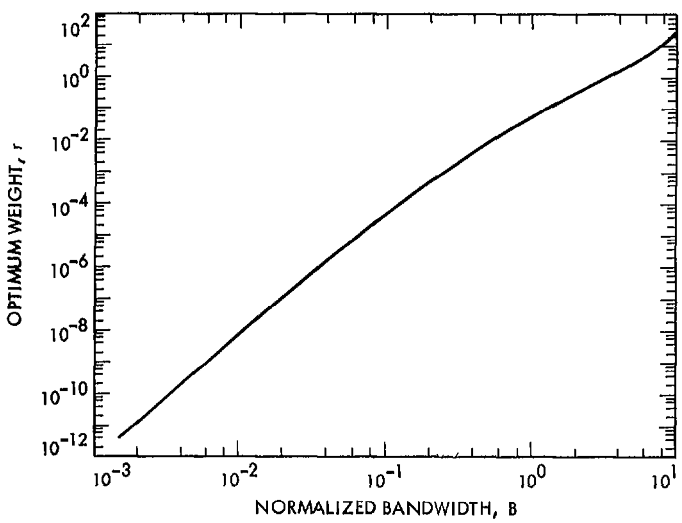

Fig. 10. Optimum weight *r* for loop filter of type II

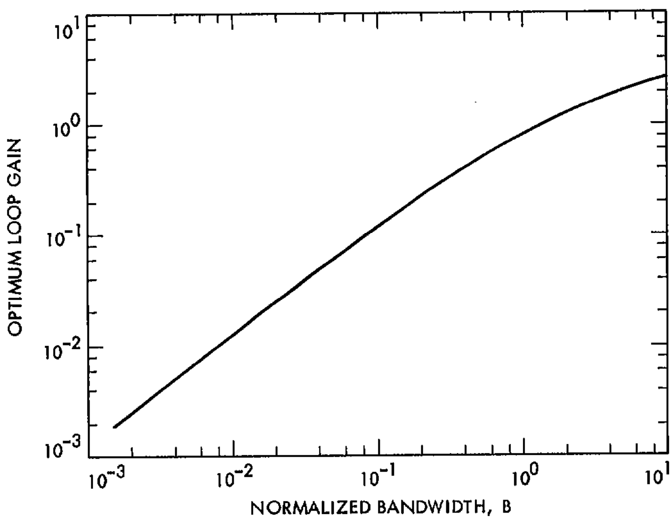

Fig. 11. Optimum loop gain vs normalized bandwidth for type II filter

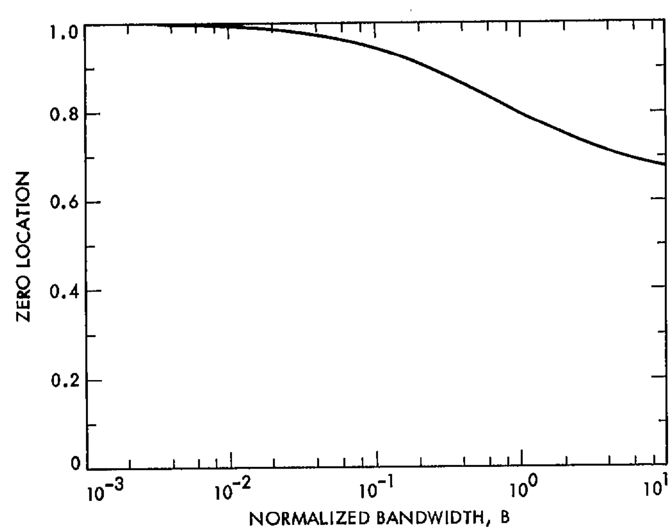

Fig. 12. Zero location of the optimum loop filter of type II

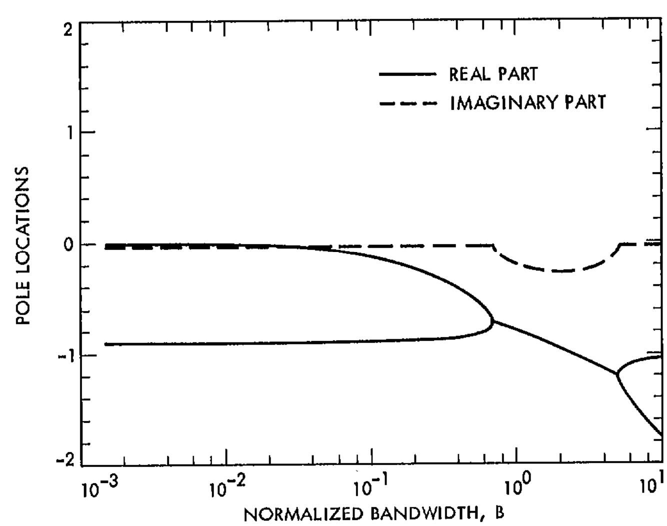

Fig. 13. Pole locations of the optimum loop filter of type II

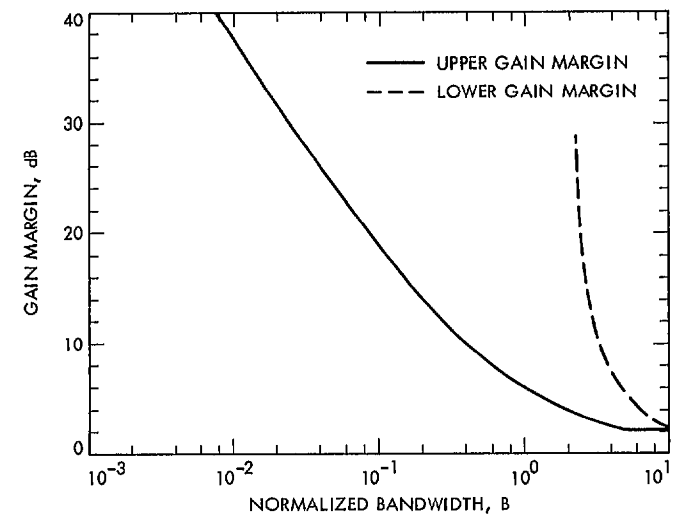

Fig. 14. Gain margin vs normalized bandwidth for type II filter

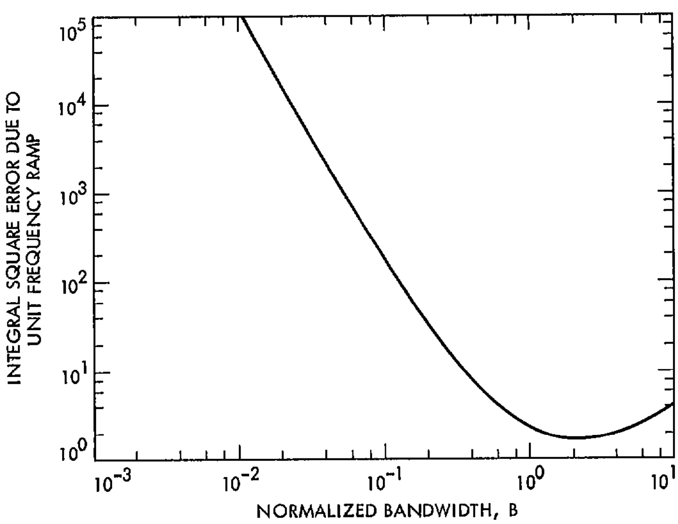

Fig. 15. Transient error performance of type II filter

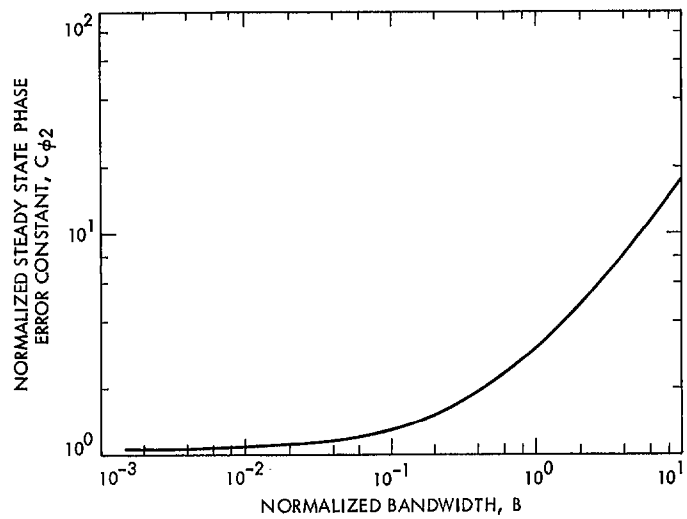

Fig. 16. Steady state error performance of type II filter

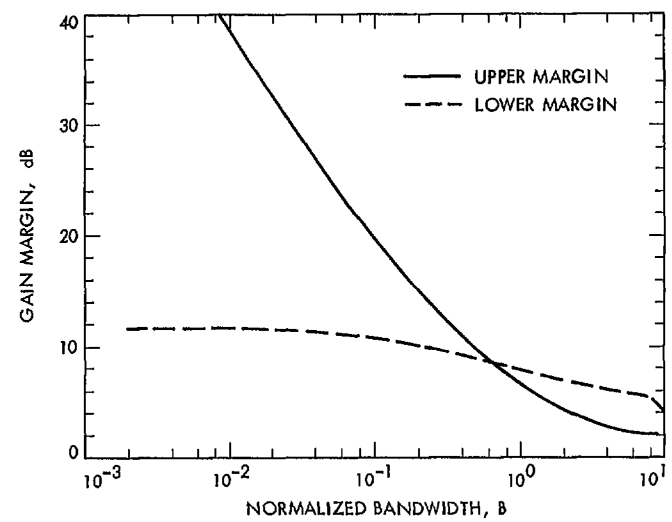

Fig. 17. Gain margin vs normalized bandwidth for type III filter

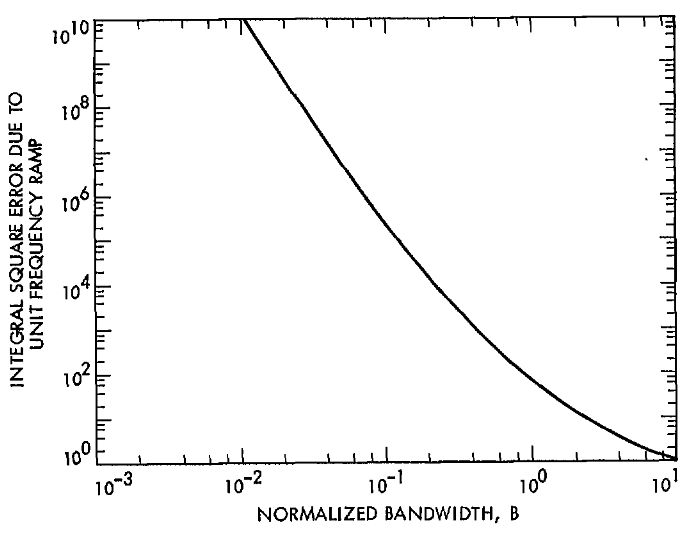

Fig. 18. Transient error performance of type III filter

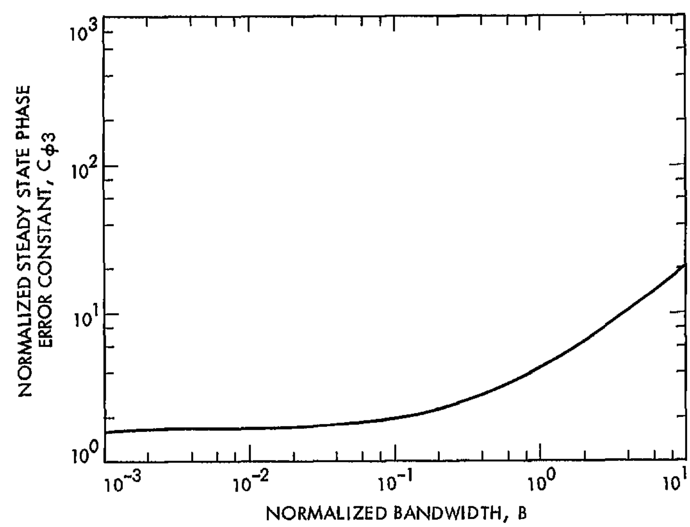

Fig. 19. Steady state error performance of type III filter

Appendix

In the following, we derive the transfer functions of the optimum filters for various phase dynamics considered in the article.

I. Case 1: Phase Step

In this case $\Theta(z) = (1 - z^{-1})^{-1}$, and the expression for $P(z)$ can be easily evaluated as,

$$P(z) = -\frac{K^2 \bar{N}_0 T^3}{4} \frac{(z+1)^2}{(z-1)^2} \left\{1 - \frac{qz}{(z-1)^2}\right\}$$

$$= -\frac{K^2 \bar{N}_0 T^3}{4} \frac{z^4 - qz^3 - 2(1+q)z - qz + 1}{(z-1)^4} \tag{A-1}$$

Factorization of $P(z)$ yields,

$$P(z) = \left\{\frac{K^2 \bar{N}_0 T^3}{4} \frac{(az^2 + bz + c)}{(z-1)^2}\right\} \times \left\{\frac{az^{-2} + bz^{-1} + c}{(z^{-1} - 1)^2}\right\} \tag{A-2}$$

where the two bracketed terms in the above represent $P^+(z)$ and $P^-(z)$ respectively. Comparing the coefficients of equal powers of z in the equivalent expressions (A-1) and (A-2), the following set of equations (referred to above as Eq. [11]) are obtained.

$$ac = -1$$

$$ab + bc = q$$

$$a^2 + b^2 + c^2 = 2(1+q)$$

The above set of equations can then be solved for unknown a, b, c.

Now,

$$\frac{\lambda N(z^{-1}) \Phi_{\Theta\Theta}(z)}{zP^-(z)} = \frac{\lambda KT}{2} \frac{(z+1)z^2}{(z-1)(a + bz + cz^2)}$$

Writing down the partial fraction expansion for the above

$$\left[\frac{\lambda N(z^{-1})\Phi_{\Theta\Theta}(z)}{zP^-(z)}\right]_+ = \frac{\lambda KT}{2} \frac{2/(a+b+c)}{(z-1)} \tag{A-3}$$

This is in view of definition of $[\ ]_+$ and the fact that the roots of $a + bz + cz^2 = 0$ are outside the unit circle. Substitution of Eq. (A-3) and the expression for $P^-(z)$ from Eq. (A-2) into Eq. (6) yields the following expression for $W_0(z)$,

$$W_0(z) = \frac{4q}{(a+b+c)} \frac{1}{KT} \frac{z(z-1)}{(az^2 + bz + c)}$$

From Eq. (7) and the expansion for $(a + b + c)^2$, it follows that $(a + b + c)^2 = 4q$, and the expression for $W_0(z)$ can be simplified to the one given in Section IV.

Straightforward substitution yields that,

$$1 - W_0(z)N(z)$$

$$= \frac{2az^3 + 2bz^2 + (c - a - b)z - (a + b + c)}{2z(az^2 + bz + c)}$$

and the optimum loop filter is given by

$$F(z) = \frac{W_0(z)}{1 - W_0(z)N(z)}$$

$$= \frac{2(a+b+c)}{KT} \times \frac{z^2(z-1)}{2az^3 + 2bz^2 + (c-a-b)z - (a+b+c)}$$

Cancellation of the factor $(z - 1)$ common to both the numerator and denominator yields Eq. (12).

II. Case 2. Frequency Step

In this case $\Theta(z) = Tz/(z-1)^2$, and the $P(z)$ is given by

$$P(z) = -\frac{K^2 \bar{N}_0 T^3}{4} \frac{(z+1)^2}{(z-1)^2} \left\{1 + r\frac{z^2}{(z-1)^4}\right\}; \qquad r = \frac{\lambda T}{\bar{N}_0}$$

$$= -\frac{K^2 N_0 T^3}{4}$$

$$\times \frac{\{z^6 - 2z^5 + (r-1)z^4 + (2r+4)z^3 + (r-1)z^2 - 2z + 1\}}{(z-1)^6}$$

Factorization of $P(z)$ results in

$$P(z) = \left\{\frac{K^2\bar{N}_0T^3}{4}\;\frac{az^3+bz^2+cz+d}{(z-1)^3}\right\}$$

$$\times\left\{\frac{az^{-3}+bz^{-2}+cz^{-1}+d}{(z^{-1}-1)^3}\right\} \qquad \text{(A-4)}$$

where the factors of $P(z)$ above represent $P^+(z)$ and $P^-(z)$ respectively. Comparison of the two alternative expressions yields the following set of equations for the solution of a, b, c and d.

$$\left.\begin{aligned} ad &= 1\\ ac+bd &= -2\\ ab+bc+cd &= (r-1)\\ a^2+b^2+c^2+d^2 &= (2r+4)\end{aligned}\right\} \qquad \text{(A-5)}$$

Now,

$$Y(z) \triangleq \frac{\lambda N(z^{-1})\,\Phi_{\Theta\Theta}(z)}{zP^-(z)}$$

$$= \frac{(z+1)z^3}{(z-1)^2}\;\frac{1}{(a+bz+cz^2+dz^3)} \qquad \text{(A-6)}$$

Writing down the partial fraction expansion of the above rational function of z and recognizing that the roots of $a+bz+cz^2+dz^3=0$ lie outside the unit circle,

$$\left[\frac{\lambda N(z^{-1})\,\Phi_{\Theta\Theta}(z)}{zP^-(z)}\right]_+ = \frac{A}{(z-1)^2}+\frac{B}{(z-1)};$$

$$A = (z-1)^2\,Y(z)\Big|_{z=1} = \frac{\lambda KT^3}{(a+b+c+d)}$$

$$B = \frac{d}{dz}(z-1)^2\,Y(z)\Big|_{z=1} = \frac{\lambda KT^3}{2}\;\frac{(7a+5b+3c+d)}{(a+b+c+d)^2}$$

Substitution of A and B, and a little simplification yields,

$$[Y(z)]_+ = \frac{\lambda KT^3}{2}\;\frac{1}{(a+b+c+d)^2}$$

$$\times\frac{\{(7a+5b+3c+d)z-(5a+3b+c-d)\}}{(z-1)^2} \qquad \text{(A-7)}$$

Substitution of Eq. (A-7) and the expression for $P^-(z)$ from Eq. (A-4) into Eq. (6) provides an expression for the optimum close loop filter $W_0(z)$,

$$W_0(z) = \left(\frac{\lambda T}{\bar{N}_0}\right)\left(\frac{2}{KT}\right)$$

$$\times\frac{1}{(a+b+c+d)^2}\;\frac{(h_0z-h_1)(z-1)z}{(az^3+bz^2+cz+d)};$$

$$h_0 = (7a+5b+3c+d)$$

$$h_1 = (5a+3b+c-d)$$

From Eq. (A-5) and the expansion for $(a+b+c+d)^2$, it follows that $(a+b+c+d)^2 = 4r$. Substitution of this in the above leads to the expression for $W_0(z)$ given in Section IV.

Straightforward substitution yields the optimum loop filter $F(z)$ as,

$$F(z) = \frac{2}{KT}$$

$$\times\frac{(h_0z-h_1)(z-1)z^2}{4az^4+4bz^3+(4c-h_0)z^2+(4d-h_0+h_1)z+h_1}$$

Factorization of the denominator polynomial and cancelling the factor $(z-1)$ common to both the numerator and denominator,

$$F(z) = \frac{2}{KT}$$

$$\times\frac{(h_0z-h_1)z^2}{(z-1)\{4az^2+(8a+4b)z+(5a+3b+c-d)\}}$$

Solution of the Filter Coefficients a, b, c, d. For any given value of r, the system of simultaneous non-linear equations, Eq. (A-5) must be solved for a, b, c and d. In fact, as discussed earlier, it is sufficient to solve the set (18) for $\bar{a}, \bar{b}, \bar{c}$ first and then obtain a, b, c, and d from

$$a = \bar{a}$$

$$b = \bar{b} + 0.9\bar{a}$$

$$c = \bar{c} + 0.9\bar{c}$$

$$d = 0.9\bar{c}$$

It turns out from the stability considerations that $|\bar{a}| > |\bar{c}|$ and without loss of any generality we may assume $a > 0$. Thus, to satisfy the first equation of (18), $\bar{a} > 1$. Since we are interested in solving the set (18) for a range of values of r, an equivalent but simpler approach is to assume a range of $\bar{a}$ and then solve for $\bar{b}, \bar{c}$, and r from the following:

$$\bar{c} = 1/\bar{a}$$

$$\bar{b} = -4/(\bar{a} + \bar{c})$$

$$r = (\bar{a}^2 + \bar{b}^2 + \bar{c}^2 - 6)^{1/2}$$